(19) We have $a^2, s \in \varphi(G)$ and so also $a \in \varphi(G)$ thus $\varphi(G) = G$.
Moreover the element

$$g = (a^{-1}s^{-1}as)^2 a^{-1} = a^{-1}s^{-1}asa^{-1}s^{-1}asa^{-1}$$

satisfies condition of Britton's lemma. Thus $g \neq 1$ in the HNN extension of $\mathbb{Z}$ (generated by a), namely $BS(2,3)$.
However

$$\varphi(g) = (a^{-2}s^{-1}a^2s)^2 a^{-2} = a^2a^{-2} = 1.$$

Thus φ surjective and not an isomorphism $\square$.

(NC) **Remark**: Say p, q are meshed if one divides the other or the set of prime divisors is the same. One proves that the Baumslag–Solitar group

$$BS(p,q) = \langle a, s \mid s^{-1}a^p s = a^q \rangle$$

is non Hopfian if p, q are not meshed.

Proof Let d be a prime dividing p and not q. The map

$\varphi(a) = a^d$, $\varphi(s) = s$ defines a homomorphism.

$$g = [a^{p/d}, s]^d a^{p-q} = a^{-p/d}s^{-1}a^{p/d}s\,a^{-p/d}s^{-1} \ldots s\,a^{p-q}$$

is non trivial by Britton's lemma.
However

$$\varphi(g) = [a^p, s]^d a^{d(p-q)} = 1$$

Thus kernel is non-trivial.

(ND) **Remark**: Let $n \neq -1, 0, 1$. Then $BS(1,n) = \langle a, b \mid a^{-1}ba = b^n \rangle$ is residually finite and hence Hopfian, as a group of matrices.
$BS(1,n) \simeq$ group generated by $A = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Proof $A^{-1}BA = B^n$. Let $w = a^{k_1}b^{l_1} \ldots a^{k_s}b^{l_s}$ arbitrary word whose value on these matrices A, B is trivial. We write it as
$w = (a^{p_1}b^{l_1}a^{-p_1})(a^{p_2}b^{l_2}a^{-p_2}) \ldots (a^{p_s}b^{l_s}a^{-p_s})\,a^{p_s}$ where

$p_i = k_1 + \dots + k_i$.

For $k > l$ $\quad a^{-k} b a^k = (a^{-l} b a^l)^{n^{k-l}}$ is a consequence of

$a^{-1} b a = b^n$. Therefore relation above is equivalent to

$w_1 = a^{-l} b^t a^l a^{p_s}$ and thus we need

$A^{-l} B^t A^l A^{p_s} = 1$

Matrix computation $\quad t = p_s = 0$ hence $w_1 = 1$. $\square$

V Another proof of Gruschko's theorem

VA Consider the graph (bouquet of circles) with a single vertex and oriented edges bijectively associated to the elements of a basis S of G. For each $s \in S$ write as above

$\varphi(s) = a_{s,1} \dots a_{s,m}$

and divide the edge associated to s into n sub-edges, each sub-edge being labeled by the corresponding $a_{s,i} \in A_{\alpha(s,i)}$. We obtain then via labeling a homomorphism

$$\varphi^*: \pi_1(K) \longrightarrow * A_\alpha$$

where K is the graph labeled; also $\varphi(e)$ = the label associated to e.

VB We add new 1-cells and 2-cells to K and keep notation K for it.

For each $i \in I$ set $K_i^0 = K^0$, $K_i^1 = \{e \in K^1;\ \varphi(e) \in A_i\}$

$K_i^2 = \{D \in K^2;\ \partial D \subset K_i^1\}$. We have then following properties:

(1) $\pi_1(K)$ and G are identified s.t. φ^* and φ are the same.

(2) $\bigcup_{i \in I} K_i = K$

(3) $K_s \cap K_t = \bigcap_{i \in I} K_i$ for $s \neq t$.

(4) $\bigcap_{i \in I} K_i$ is a disjoint union of trees.

We will modify K so that the new complex K fulfills (1–4) above and also condition

(5) $\bigcap_{i \in I} K_i$ is connected (and hence a tree).

Suppose that this $\bigcap_{i \in I} K_i$ is NOT connected.

(2)

(VC) $\boxed{\text{Def.}}$ $\boxed{\text{A binding tie is a path } p \text{ in } \underline{\text{~~some~~ } K_j} \text{ connecting vertices in different components of } \bigcap_{i \in I} K_i \text{ such that } \varphi^*(p) = 1}$

Assume that there exists a binding tie p. We add to K a new edge e (with the same initial and final ~~ed~~ vertices ~~than~~ as p) and a new 2-cell with boundary pe^{-1}. Extend φ^* ~~by~~

$$\varphi^*(e) = 1.$$

Then the new complex K has one connected component fewer than the older and still verifies (1–4).

Therefore in finitely many steps we will arrive to a complex K for which $\bigcap_{i \in I} K_i$ is connected and thus a tree.

Now $K^0 \subset \bigcap_{i \in I} K_i$ and so K_i is connected $\forall i \in I$.

Using Van Kampen thm $\pi_1(K) = \ast_{i \in I} \pi_1(K_i)$. Let $G_i = \pi_1(K_i)$. Then $G = \ast_{i \in I} G_i$ and $\varphi(G_i) \subset A_i$. Since φ is surjective $\Rightarrow \varphi(G_i) = A_i$. This ends the proof, but we have to prove the existence of binding ties.

(VD) $\boxed{\text{Lemma}}$ $\boxed{\text{If } \bigcap_{i \in I} K_i \text{ is not connected then a binding tie exists.}}$

<u>Proof of lemma</u>: Let v a vertex of a component of $\bigcap_{i \in I} K_i$ not containing the vertex x of the initial bouquet of circles.

Let p be a path in K joining v to ~~x~~.

Since φ is surjective there exists a closed path q in K starting at x such that $\varphi^*(q) = \varphi^*(p)$.

Thus $q^{-1}p$ goes from x to v and $\varphi^*(q^{-1}p) = 1$

We have $\bigcup_{i \in I} K_i = K$ and thus $q^{-1}p = r_1 \dots r_k$ where r_j are maximal paths lying in the same $K_{\sigma(j)}$. Thus

$r_j \subset K_{\sigma(j)}$ and r_j, r_{j+1} do not lie in the same $K_{\sigma(j)}$

Now $1 = \varphi^*(r) = \varphi^*(r_1)\dots\varphi^*(r_k)$. $\varphi^*(r_j) \in G_{\sigma(j)}$;
This cannot be a reduced word in $\underset{j\in I}{*} G_j$ and thus there is some s with

$$\varphi^*(r_s) = 1.$$

If endpoints of r_s lie in different components of $\bigcap_{i\in I} K_i$ then r_s is a binding tie. Suppose they are in the same component. Choose then ~~r_s~~ r_s' a path in $\bigcap_{i\in I} K_i$ joining endpoints of r_s.
Then $\varphi^*(r_s') = 1$ and we can replace in r the path r_s by r_s'. Now define $r_{s-1}' = r_{s-1} r_s'$. Since

$r_s' \subset \bigcap_{i\in I} K_i \Rightarrow r_{s-1}' \subset K_{\sigma(s-1)}$. Thus we

reduced $r = r_1 \dots r_{s-2} r_{s-1}' r_{s+1} \dots r_k$, to a product of $(k-1)$ factors. Using this procedure we will eventually find a binding tie. $\square$

VI EXERCICES.

(1) A, B finitely presented groups. Then $A \underset{H}{*} B$ is finitely presented, for $H \hookrightarrow A$, $H \hookrightarrow B$ iff H is finitely generated.

(2) $H, K \subset B$ isomorphic groups $\theta : H \to K$ an isomorphism, B finitely presented. Then $B *_\theta = \langle B, t \mid tHt^{-1} = K \rangle$ is finitely presented iff H is finitely generated.

(3) The group generated by $(a_1, a_2), (1, b_2), (b_1, 1)$ in $\mathbb{F}(a_1, b_1) \times \mathbb{F}(a_2, b_2)$ is not finitely presented. Actually this has a presentation

$$\langle \alpha, \beta, \gamma \mid [\beta, \alpha^i \gamma \alpha^{-i}] = 1, \ \forall i \in \mathbb{Z} \rangle.$$