Exercices

1. Prove that $\mathbf{Z/pZ} *_{\mathbf{Z}} \mathbf{Z/qZ} = \mathbf{1}$ if $gcd(p,q) = 1$. Moreover, show that $\mathbf{Z/pZ} *_{\mathbf{Z}} \mathbf{Z/pZ} \neq \mathbf{1}$. Here, the homomorphisms $\mathbf{Z} \to \mathbf{Z/pZ}$ and $\mathbf{Z} \to \mathbf{Z/qZ}$ are the mod p (respectively mod q) reduction.

2. Show that $\mathbf{Z/2Z} * \mathbf{Z/2Z}$ has a free normal group of finite index although it is not a free group. Notice that a deep theorem of Stallings states that a torsion free group which admits a free subgroup of finite index is itself free.

3. Find a presentation for the fundamental group of the closed non-orientable surface of genus n.

4. Let G denote the iterated free product $\mathbf{Z/2Z} * \cdots * \mathbf{Z/2Z}$. Find the minimal number of generators of a group presentation of G (called the rank $rk(G)$ of G). Show that if we have an exact sequence of groups

$$1 \to K \to G \to H \to 1$$

then $rk(G) \leq rk(H) + rk(K)$. Give an example of an exact sequence for which the inequality above is strict.

5. Although $\mathbf{Z/2Z} * \mathbf{Z/3Z}$ is not a free group it admits a nontrivial action on a tree. Find explicitly such an action and explain.