Exercices

1. Let Γ_z denote the subgroup of $SL(2,\mathbf{C})$ generated by $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$.

(1) If $|z| \geq 2$ then Γ_z is free.
(2) Find other values of z for which Γ_z is free.
(3) If $z = \frac{1}{k}$, $k \in \mathbf{Z}$ then Γ_z is not free.
(4) If $q^2 - Np^2 = 1$, where p, q, N are nonzero integers then $\Gamma_{p/q}$ is not free.

2. Let D_1, D_2, D_3, D_4 be disjoint closed disks in $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.

(1) Show that there exists an element $\gamma \in PSL(2,\mathbf{C})$ such that γ maps the exterior of D_1 onto the interior of D_2.
(2) Let δ be an element of $PSL(2,\mathbf{C})$ mapping the exterior of D_3 onto the interior of D_4. Prove that the subgroup generated by γ and δ is free.
(3) Generalize to an arbitrary (even) number of disks.

3. Let X be a tree. An automorphism γ of X has an inversion if there exists an edge $e = vw$ of X such that $\gamma(v) = w, \gamma(w) = v$. An automorphism γ is said to be hyperbolic if it has no inversions and its amplitude $a_\gamma = \min_{x \in X} d(x, \gamma(x))$ is strictly positive.

(1) Consider henceforth γ hyperbolic. Prove that $\{x \in X; d(x,\gamma(x)) = a_\gamma$ is the set of vertices of a geodesic line in X, called the axis of γ.
(2) Let γ_1, γ_2 be hyperbolic automorphisms of axes L_1 and L_2. If $L_1 \cap L_2$ is empty or one vertex then the subgroup generated by γ_1 and γ_2 is free.
(3) If $L_1 \cap L_2$ has length l then the subgroup generated by $\gamma_1^{m_1}$ and $\gamma_2^{m_2}$ is free, if $m_i a_i > l$, $i = 1, 2$.