Exercices

1. Let X be a tree. An automorphism γ of X has an inversion if there exists an edge $e = vw$ of X such that $\gamma(v) = w, \gamma(w) = v$. An automorphism γ is said to be hyperbolic if it has no inversions and its amplitude $a_\gamma = \min_{x \in X} d(x, \gamma(x))$ is strictly positive.

   (1) Consider henceforth γ hyperbolic. Prove that $\{x \in X; d(x, \gamma(x)) = a_\gamma$ is the set of vertices of a geodesic line in X, called the axis of γ.

   (2) Let γ_1, γ_2 be hyperbolic automorphisms of axes L_1 and L_2. If $L_1 \cap L_2$ is empty or one vertex then the subgroup generated by γ_1 and γ_2 is free.

   (3) If $L_1 \cap L_2$ has length l then the subgroup generated by $\gamma_1^{m_1}$ and $\gamma_2^{m_2}$ is free, if $m_i a_i > l$, $i = 1, 2$.

2. Let $p \geq 2$ be a fixed integer and identify S^1 as $\mathbf{R/4pZ}$. Let $f_0 \in PL(S^1)$ be the piecewise-linear homeomorphism of S^1 sending linearly $[1, p]$ to $[p, 2p-1]$, $[p, 3p]$ to $[2p-1, 2p+1]$, $[3p, 4p-1]$ to $[2p+1, 3p]$ and $[4p-1, 1]$ to $[3p, p]$. Let R_a denote the rotation carrying 0 to a. and define $f_a = R_{2a}^{-1} f \circ f_0 \circ R_{2a}$, where $a = 0, 1, 2, ..., 2p-1$. Prove that $f_0^2, f_1^2, ..., f_{p-1}^2$ generate a free subgroup of $PL(S^1)$.